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Invariantive Classification of Pairs of Conics Modulo 2.

By L. E. Dickson.

§ 1. *Relation to a Generalization of Recent Problems on Point Sets.*

The projective classification of sets of points in a modular space has been utilized in several recent papers by the writer to effect a complete projective classification of modular curves or surfaces of given orders for low moduli. Finite sets of points in an ordinary space have been investigated quite recently by Coble.† For either modular or ordinary space, we therefore have information also concerning sets of elements (lines or planes) dual to points. But a new situation arises if we consider systems composed of points and lines (or planes). To such a problem we can at once reduce the problem of the classification of one or more conics modulo 2. In fact with such a conic

$$F=a_1yz+a_2xz+a_3xy+b_1x^2+b_2y^2+b_3z^2$$

is associated covariantly‡ an apex $A=(a_1, a_2, a_3)$ and a line

$$L=(\beta_1+1)x+(\beta_2+1)y+(\beta_3+1)z,$$

where

$$\beta_1=b_1+\alpha_2\alpha_3, \quad \beta_2=b_2+\alpha_1\alpha_3, \quad \beta_3=b_3+\alpha_1\alpha_2 \qquad (\alpha_i=a_i+1).$$

Conversely, A and L evidently uniquely determine a conic F.

When F is a double line, the a's are congruent to zero and there is no apex. When F is a pair of imaginary lines, the coefficients of L are all congruent to zero ("Colloq.," p. 76, table). The converse of each statement is true.

Hence *the projective classification of pairs of conics* F, F' *modulo* 2 *is equivalent to the projective classification of the systems* A, L, A', L' *of two points and two lines modulo* 2, *and the degenerate systems with fewer points or fewer lines.*

§ 2. *Geometrical Classification of Pairs of Conics.*

For the present, let the conics have apices A and A' which are distinct and lines L and L' which are distinct.

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† *Trans. Amer. Math. Soc.*, Vol. XVI (1915).

‡ Dickson, "Madison Colloquium Lectures," 1914, p. 69, cited as "Colloq."

First, let neither A nor A' be on L or L'. Since there are only three real points on $l=AA'$, the intersection P of L and L' is on l. After a real transformation, we have $A=(001)$, $P=(011)$, $L=y+z$. Then A' is the third real point (010) on $l(x=0)$, and L' is the third real line $x+y+z=0$ through P. Thus $F=xy+x^2+z^2$, $F'=xz+y^2$.

Second,* let neither A nor A' be on L', while A', but not A, is on L. After a transformation, we have $l=x$, $L=x+z$, $L'=x+y+z$, whence $A'=(010)$. Since the intersection of l and L' is (011), A is the remaining real point (001) on l. Thus $F=xy+y^2+z^2$, $F'=xz+y^2$.

Third, let A', but not A, be on L; A, but not A', on L'. We may take $L=x+z$, $L'=x+y$, $l=x$. Then $F=xy+y^2+z^2$, $F'=xz+y^2+z^2$.

Fourth, let A be on L, A' on L'. If A is on L', we may take $L=x+y$, $L'=l=x$, $A'=(010)$. Then $F=xy$, $F'=xz+z^2$. But, fifth, if A is not on L' and A' is not on L, we may take $l=x$, $L=x+y$, $L'=x+z$, whence $F=xy$, $F'=xz$.

Sixth, let A be on L, A' not on L'. If A is not on L' and A' not on L, we may take $l=x$, $L=x+y$, $L'=x+y+z$. Then $A'=(010)$, $F=xy$, $F'=xz+y^2$. But, seventh, if A is on L', and A' not on L, we may take $L=x+y$, $A=(001)$, $A'=(010)$, whence $L'=y$, $F=xy$, $F'=xz+x^2+y^2+z^2$. Again, eighth, if A is not on L', but A' on L, we may take $A=(001)$, $L=x$, $L'=x+y+z$, whence $A'=(010)$, $F=xy+y^2$, $F'=xz+y^2$. Ninth, if A is on L', and A' on L, we may set $L=x$, $L'=y$, $A'=(010)$, whence $F=xy+y^2$, $F'=xz+x^2+y^2+z^2$.

Next, if there are not two distinct apices (including the cases of an absence of one or both apices) and not two distinct lines L and L', two configurations of the same type are evidently projectively equivalent. The types of conics will not be listed. The same result was seen to hold in the above nine cases.

Two pairs of conics modulo 2 *are projectively equivalent if and only if they have the same properties as regards existence of apices and covariant lines, distinctness of apices and lines, and incidence of apices and lines.*

§ 3. *Modular Invariants of a Pair of Conics.*

Since $P=\alpha_1\alpha_2\alpha_3$ is 0 or 1 according as F has an apex or not, it is an invariant of F modulo 2. Similarly, we have the following invariants of F and F': $P'=\alpha'_1\alpha'_2\alpha'_3$; $J=\beta_1\beta_2\beta_3$, with the value 0 or 1 according as L is not or is identically zero; $J'=\beta'_1\beta'_2\beta'_3$; $\Delta_1=\Sigma(\beta_i+1)a_i$, which is 0 or 1 according as* A is on L or not on L; Δ'_1 for F'; $\delta_1=\Sigma(\beta_i+1)a'_i$, which is 0 or 1 according as A' is on or not on L; δ'_1 for A on L';

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* The companion case given by interchanging accented and unaccented letters will not be treated separately. Similarly throughout.

$$v=\prod_{i=1}^{3}(1+a_i+a'_i),\quad \beta=\prod_{i=1}^{3}(1+\beta_i+\beta'_i),$$

where $v=0$ or 1 according as A and A' are distinct or coincident, while $\beta=0$ or 1 according as L and L' are distinct or identical.

In view of § 2 and a general theorem ("Colloq.," p. 14), *these invariants form a fundamental system of invariants of F and F' modulo* 2.

§ 4. *Certain Formal Invariants of F and F' Modulo* 2.

When the coefficients of F and F' are independent variables, instead of integers taken modulo 2 as hitherto, the invariants under all linear transformations with integral coefficients taken modulo 2 are called formal invariants.

Since a_1, a_2, a_3 are cogredient with x, y, z,

$$\Delta=F(a_1,\,a_2,\,a_3)=a_1a_2a_3+a_1^2b_1+a_2^2b_2+a_3^2b_3$$

is a formal invariant of F. If we set

$$B_1=b_1^2+a_2a_3+a_2^2+a_3^2,\ \ldots,\ B_3=b_3^2+a_1a_2+a_1^2+a_2^2,$$

we find that ("Colloq.," p. 70)

$$L=B_1x+B_2y+B_3z$$

is a formal covariant of F, which reduces to L of § 1 for integral coefficients taken modulo 2. Hence

$$\Delta_1=\Sigma a_iB_i,\quad \Delta'_1=\Sigma a'_iB'_i,\quad \delta_1=\Sigma a'_iB_i,\quad \delta'_1=\Sigma a_iB'_i$$

are formal invariants of F and F'. From the universal covariant

$$Q_1=\Sigma x^4+\Sigma x^2y^2+xyz\Sigma x,$$

we obtain the formal invariants of F

$$C=\Sigma a_1^4+\Sigma a_1^2a_2^2+a_1a_2a_3\Sigma a_1,\quad E=\Sigma B_1^4+\Sigma B_1^2B_2^2+B_1B_2B_3\Sigma B_1,$$

which reduce, for integral values of a_i, b_i, to $1+P$ and $1+J$ of § 2. The remaining two universal covariants, which form with Q_1 a fundamental system ("Colloq.," p. 76), vanish for integral values. Every formal invariant of F of degree <4 is a linear function of Δ, Δ_1, and

$$\alpha=\Sigma a_ib_i+\Sigma a_i^2+a_1a_2+a_1a_3+a_2a_3.$$

To form intermediate formal invariants, we construct the same function of $F+kF'$ that a given invariant is of F. Thus Δ_1 and Δ become

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* In this sentence, A denotes the set a_1, a_2, a_3, whether or not all the a's are zero.

$$\Delta_1+k\delta_1+k^2\delta_1'+k^3\Delta_1', \quad \Delta+k\delta+k^2\delta'+k^3\Delta',$$

where

$$\delta=a_1a_2a_3'+a_1a_3a_2'+a_2a_3a_1'+a_1^2b_1'+a_2^2b_2'+a_3^2b_3'$$

vanishes if and only if vertex A of F is on F', while δ' vanishes if and only if A' is on F. From C we get

$$C+k\Sigma a_1^2(a_2a_3'+a_2'a_3)+k^2C_2+k^3\Sigma a_1'^2(a_2a_3'+a_2'a_3)+k^4C',$$

$$C_2=\sum_3(a_1^2a_2'^2+a_1'^2a_2^2+a_1^2a_2'a_3'+a_1'^2a_2a_3).$$

For integral a_i, b_i, the coefficients of k and k^3 vanish, while

$$C_2\equiv v+P+P'+1 \pmod 2,$$

so that, if the conics have apices, the apices coincide if and only if $C_2\equiv 0$.

The formal invariant intermediate between α and α' is

$$\rho=\Sigma(a_ib_i'+a_i'b_i+a_i'a_j).$$

For integers, $\rho\equiv\delta+\delta_1\equiv\delta'+\delta_1' \pmod 2$. If in C_2 we replace the a's by B's, we get the formal invariant

$$\sigma=\Sigma B_1^2B_2'^2+\Sigma(B_1^2B_2'B_3'+B_1'^2B_2B_3).$$

For integers, $\sigma\equiv\beta+J+J'+1 \pmod 2$. From E we get

$$E+kE_1+k^2E_2+k^3E_3+k^4E_4+k^5E_3'+k^6E_2'+k^7E_1'+k^8E',$$

where, if $c_{ij}=a_ia_j'+a_i'a_j$,

$$E_1=\Sigma c_{12}B_1^2B_2,$$

$$E_2=\Sigma(c_{12}^2B_1^2+c_{12}c_{13}B_1^2+c_{23}^2B_2B_3+B_1^2B_2B_8'),$$

$$E_3=\Sigma(c_{12}B_1^2B_2'+c_{12}c_{23}^2B_2),$$

$$E_4=\sigma+\Sigma c_{23}^2(c_{23}^2+c_{13}^2+c_{12}c_{13}+B_2B_3'),$$

where the sums are in the sense of symmetric functions. Evidently E_i' is derived from E_i by interchanging accented and unaccented letters.